A Model To Deal With Market Impact When Executing Transactions

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Summary

Introduction

For several decades, financial mathematics have represented one of the important fields of applied mathematics, and this field can itself be divided into several areas. The most famous of those areas is undoubtedly the pricing of derivatives and options, for instance via the Black-Scholes formula. Nonetheless there exist plenty of other areas [1] within financial mathematics: even though they can seem, at first sight, less "appealing" than pricing topics, they remain crucial and play a pivotal role when it comes to implementing financial strategies in real life. In this paper we choose to dwell on one of these key areas of financial mathematics: the questions raised by the execution of given transactions.

Financial execution is indeed of a paramount importance: the very existence of finance itself is based on the fact that a huge variety of people execute every day thousands and thousands of orders, thus giving birth to what is known as a financial market. But executing an order is not as direct as it sounds insofar as many steps are required, and the way the execution is realized may have a financial impact.

Market impact is arguably the simplest way to understand how execution may have such consequences. Indeed when a given investor decides to sell a certain amount of a given stock, the way he or she does it may affect how the price of the stock moves. Therefore the execution itself has to be taken into account when an investment decision is made. This leads us to develop in this paper a mathematical model to represent the price of an asset in presence of market impact caused by the action of a single trader. Based on this model we can solve the problem of execution a trader faces when he or she decides to sell a certain amount of shares.

1 Understanding How A Financial Market Works To Understand How Market Impact May Appear

Before delving into the mathematical considerations of our model, it is worth paying a little attention to how a financial market works practically.

Indeed the mathematical approach of financial markets is mainly premised on the modeling of the price of an asset: the purpose is to construct a model which faithfully reproduces some of the stylized facts which are observed in real life. A model is deemed helpful if it captures the most important stylized facts. This is exactly the kind of approach which was chosen when mathematicians first tried to price options.

Nonetheless such an approach lends only little importance to the way a price is determined on a given market. In real life, a market is merely an order book: when people decide to sell or buy a given asset, they propose a quantity Q and a price P: they are ready to buy or sell Q shares of an asset for a price of P per share. All the relevant information is then gathered in an order book.

Since in this first part, we insist on the practical side of financial markets, we decide to work with a real example. Let us consider an asset, for instance the cryptocurrency bitcoin; we deliberately choose an intriguing asset instead of a more common one such as a stock, in order to show that the staples of a market are exactly the same from one market to another.

At a given date t, for instance the 12th of November 2019, at 11:25pm, we have the following order book:

Prix(TUSD)	Montant(BTC)	Total(TUSD)	Prix(TUSD)	Montant(BTC)	Total(TUSD)
8773.64	0.683949	6,000.72230436	8749.39	0.002284	19.98360676
8773.28	0.088718	778.34785504	8746.93	0.032480	284.10028640
8771.60	0.196000	1,719.23360000	8745.77	0.082462	721.19368574
8770.15	0.684139	6,000.00165085	8745.76	1.143411	9,999.99818736
8766.65	0.684435	6,000.20209275	8745.47	0.002485	21.73249295
8766.28	0.222000	1,946.11416000	8745.28	0.026374	230.64801472
8766.27	0.673100	5,900.57633700	8743.16	0.006935	60.63381460
8763.16	0.684745	6,000.52999420	8743.00	0.079775	697.47282500
8762.68	0.054608	478.51242944	8742.55	0.100000	874.25500000
8760.87	0.228482	2,001.70109934	8736.12	0.228934	1,999.99489608
8759.91	0.212000	1,857.10092000	8736.11	0.152000	1,327.88872000
8759.81	0.013525	118.47643025	8733.00	0.011450	99.99285000
8759.65	0.228319	1,999.99452835	8732.00	0.011452	99.99886400
8759.64	0.150000	1,313.94600000	8731.00	0.011453	99.99614300
8756.17	0.685314	6,000.72588738	8730.00	0.011454	99.99342000
8755.77	0.027280	238.85740560	8729.00	0.011456	99.99942400
8755.66	0.073926	647.27092116	8728.09	0.001298	11.32906082
8754.02	0.050000	437.70100000	8728.00	0.011457	99.99669600

Figure 1: Order book of the bitcoin market at a given date t

The left part of the chart displays the selling orders. At each line, we can read both a price and a quantity; for instance the last line means that, at t, some investors are ready to sell a quantity of 0.05 bitcoins for a price of 8754.02 dollars per bitcoin. This price is the ask price at t.

The right part of the chart displays the buying orders. Again, at each line, we can read a price as well as a quantity. The idea is very similar: the first line means that, within the bitcoin market at t, some investors are ready to buy 0.002284 bitcoins for a price of 8749.39 dollars per bitcoin. This price is the bid price at t.

When a match appears between a selling order and a buying one, a transaction happens. For example, if investor A decides at t to send a buying order for 0.01 bitcoin with a maximum price per bitcoin of 8754.02 dollars, he or she will deplete the first selling order by 0.01 bitcoin.

A first lesson can be drawn from the example: execution may require a certain amount of time. Whenever an investor decides to sell a quantity of asset, he or she can either decide to sell at a lower price by accepting the prices proposed by buyers – the transaction is then realized instantaneously – or set a price which is higher than the first buying price. In this case, the transaction is not immediate.

Furthermore we can draw a second lesson from figure 1. Even if it sounds obvious, it is important to bear in mind that execution is what shifts the price. Let us imagine that an investor A wants to buy immediately an important amount of bitcoins, for instance 2 bitcoins. By doing so he or she is going to deplete the selling orders, from bottom to top. Indeed:

$$0.05 + 0.073926 + 0.027280 + 0.685314 + 0.15 + 0.228319 + 0.013525$$
$$+0.212 + 0.228482 + 0.054608 = 1.693454$$

and

$$0.05 + 0.073926 + 0.027280 + 0.685314 + 0.15 + 0.228319 + 0.013525$$
$$+0.212 + 0.228482 + 0.054608 + 0.684745 = 2.378199$$

Investor A is going to exhaust the first eight selling orders, and then consume part of the ninth selling order. After his or her intervention, the first selling order will be: 8763.16 for a quantity of 0.378199 bitcoins. As we see, the ask price goes up, from 8754.02 dollars to 8763.16 dollars; the price is affected by A's action. This leads us to define market impact:

Definition 1 (Market Impact)
Market impact is the effect produced by a market participant when it buys or sells an asset. Such an impact may appear if the market is not liquid enough, or if the market participant intervenes massively by selling or buying a huge amount of asset.

The purpose of our paper is to devise a mathematical model of price which takes into account the market impact produced by a single trader. If it is possible to construct mathematical models [2] [3] which precisely take into account all the events modifying the order book, we choose a more "macro" approach, where the evolution of the price is what we focus on. It is then possible to use our model to mathematically work out the question of investment in presence of market impact.

2 A Simple Mathematical Model Of Price With Market Impact

In order to set forth our model, we assume a simple situation: on a given stint denoted $[0, T]$, a trader has to sell an important amount of a certain stock. The model's purpose is to take into account the market impact on the stock price generated by the selling decisions made by the trader. If the trader decides not to sell any shares at a given date, such a decision does not modify the observed price; but if the trader decides to sell some shares, this specific action should have a negative effect, be it permanent or temporary, on how the price moves.

Notations
We denote X the number of shares which are to be sold. The whole sale has to be completed before a date denoted T. The trader can only intervenes at N predetermined discrete times, based on a time step τ; thus we have

$$\tau = \frac{T}{N}$$

The predetermined times are denoted $t_k = k\tau$ for $0 \leq k \leq N$. Therefore we consider $N + 1$ times:

$$0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$$

The strategy chosen by the trader can then be seen as a vector of size $N + 1$:

$$\vec{x} = (x_0, x_1, \ldots, x_N)^T \in \mathbb{R}^{N+1}$$

where x_i denoted the number of remaining shares in the trader's portfolio at t_i. By definition $x_0 = X$ and $x_N = 0$.

The number of shares sold by the trader between two consecutive times t_{k-1} and t_k, where $1 \leq k \leq N$, is given by:

$$n_k = x_{k-1} - x_k$$

We have now all the elements to define a price which takes into account the market impact generated by the trader's actions.

Definition 2 (Price)
The price of the considered asset at time t_k with $0 \leq k \leq N$ is defined as follows: at 0, the price is known and is denoted S_0; at t_k with $1 \leq k \leq N$, the price S_k is given by the formula:

$$S_k = S_{k-1} + \sigma\sqrt{\tau}\xi_k - \tau g\left(\frac{n_k}{\tau}\right)$$

where σ is a mere volatility parameter. The random variables ξ_k are independent and identically distributed, with $\mathbb{E}[\xi_k] = 0$. g is an increasing non-negative function such that $g(0) = 0$ which represents the permanent market impact.

The most important point in the above definition is function g, which accounts for the permanent market impact. The action of the trader between t_{k-1} and t_k has an impact on all the subsequent prices S_q with $k \leq q$.

If the trader decides, for instance, to sell a huge amount of shares between t_{k-1} and t_k, then n_k is important. Such an action should result in a negative impact on the price of the asset.

On the contrary, if the trader decides not to sell any action between t_{k-1} and t_k, then $n_k = 0$ and the price should not be affected by n_k.

This reasoning justifies why g needs to be an increasing, non-negative function such that $g(0) = 0$. The modeling would not be realistic otherwise.

Nonetheless market impact is not solely permanent; it seems rather obvious that there exists a temporary market impact: when a trader sells some shares, due the inherent working of the order book, the price will necessary go down. But this downward moves does not necessarily have consequences on the long-term price of the asset. To model this short-term phenomenon, we define a new price, called the modified price:

Definition 3 (Modified Price)
The modified price at time t_k for $1 \leq k \leq N$, denoted S_k^m, is defined as follows:

$$S_k^m = S_{k-1} - h\left(\frac{n_k}{\tau}\right)$$

where h is a function which represents the temporary market impact which stems from the sales made between t_{k-1} and t_k

by the trader. The model is consistent with reality when h is an increasing function.

The modified price is the most salient point of our model; this is indeed the price the trader faces when he or she decides to sell some shares at t_k.

3 The Problem Of Optimal Execution

The above-mentioned model directly leads us to consider the so-called optimal execution problem: what is the best selling strategy the trader can implement if he or she wants to take into account the market impact induced by his or her decisions?

The problem can be solved using a mean-variance framework: the trader is going to maximize the strategy's profit while keeping the strategy's variance under a certain value. This value depends on the level of risk the trader is ready to accept.

Before positing the mathematical definitions of the problem, we remind the reader that an investment strategy is merely a vector $\vec{x} \in \mathbb{R}^{N+1}$, with $x_0 = X$ and $x_N = 0$.

Definition 4 (Implementation Shortfall)
The wealth of the trader at 0 is known: it is worth $X \times S_0$. If the trader follows a strategy $\vec{x}$, his or her wealth at T is equal to:

$$\sum_{i=1}^{N} n_i \times S_i^m$$

because, for each period $[t_{i-1}, t_i)$ with $1 \leq i \leq N$, the trader sells $n_i = x_i - x_{i-1}$ shares for a price S_i^m.
The implementation shortfall for the strategy $\vec{x}$, denoted $IS(\vec{x})$ is equal to the difference between the initial wealth and the final one:

$$IS(\vec{x}) = XS_0 - \sum_{i=1}^{N} n_i S_i^m$$

Mathematically $IS(\vec{x})$ is a random variable, so it is possible to compute its expectation and its variance.

Proposition 1 (Implementation Shortfall Expectation)
The expectation of the implementation shortfall for a given strategy $\vec{x}$ is worth

$$\mathbb{E}\left[IS(\vec{x})\right] = \sum_{k=1}^{N} x_k \tau g\left(\frac{n_k}{\tau}\right) + \sum_{k=1}^{N} n_k h\left(\frac{n_k}{\tau}\right)$$

The proof is quite simple:

$$\sum_{i=1}^{N} n_i S_i^m = \sum_{i=1}^{N}(x_{i-1} - x_i)S_{i-1} - \sum_{i=1}^{N} n_i h\left(\frac{n_i}{\tau}\right)$$

$$= \sum_{i=0}^{N-1} x_i S_i - \sum_{i=1}^{N} x_i S_{i-1} - \sum_{i=1}^{N} n_i h\left(\frac{n_i}{\tau}\right)$$

Since $x_N = 0$

$$= XS_0 + \sum_{i=1}^{N} x_i(S_i - S_{i-1}) - \sum_{i=1}^{N} n_i h\left(\frac{n_i}{\tau}\right)$$

By taking the expectation, we easily get the desired result.

Proposition 2 (Implementation Shortfall Variance)
The variation of the implementation shortfall for a given strategy $\vec{x}$ is worth

$$\mathbb{V}\left[IS(\vec{x})\right] = \sigma^2 \sum_{k=1}^{N} \tau x_k^2$$

The spirit of the proof is similar to the expectation's one. We do not mention all the details.

The trader's problem of optimal execution can then be stated as follows:

Definition 5 (Optimal Execution Problem)
Depending on his or her risk aversion, the trader chooses a maximum level of variance, denoted v_M, for the strategy. The optimal execution problem can be written:

$$\min_{\vec{x} \in \mathbb{S}} \mathbb{E}\left[IS(\vec{x})\right]$$

where

$$\mathbb{S} = \left\{\vec{x} \in \mathbb{R}^{N+1} \middle| \, x_0 = X, \; x_N = 0, \; \mathbb{V}\left[IS(\vec{x})\right] \leq v_M\right\}$$

To solve this problem mathematically, we are going to use a Lagrangian approach. The Lagrangian of this problem is the following:

$$L(\vec{x}, \lambda) = \mathbb{E}\left[IS(\vec{x})\right] + \lambda \left\{\mathbb{V}\left[IS(\vec{x})\right] - v_M\right\}$$

where $\lambda \geq 0$. With this formulation, λ can be directly identified as the trader's risk aversion factor. If $\lambda = 0$, the trader pays no attention to the risk involved by the strategy: he or she only looks for the lower expectation shortfall, whatever the risk of the strategy is. On the contrary, if $\lambda \to \infty$, the trader is extremely averse to risk and prefers a certain result, even if he has to lose money.

The optimal execution problem can be explicitly solved for simple functions g and h. As of now we consider:

$$g(u) = \gamma u$$

and

$$h(u) = \epsilon sgn(u) + \eta u$$

with $\gamma > 0$, $\eta > 0$ and $\epsilon > 0$.

The expectation of the implementation shortfall becomes:

$$\mathbb{E}\left[IS(\vec{x})\right] = \frac{1}{2}\gamma X^2 + \epsilon \sum_{k=1}^{N} |n_k| + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^{N} n_k^2$$

where

$$\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$$

Proof: we have

$$\mathbb{E}\left[IS\left(\vec{x}\right)\right] = \sum_{i=1}^{N} x_i \gamma n_i + \epsilon \sum_{k=1}^{N} |n_k| + \sum_{k=1}^{N} \frac{n_k^2}{\tau}\left(\eta - \frac{1}{2}\gamma\tau + \frac{1}{2}\gamma\tau\right)$$

$$= \sum_{k=1}^{N} n_k^2 \frac{\tilde{\eta}}{\tau} + \frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2 + \frac{\gamma}{2}\sum_{i=1}^{N} 2x_i n_i + \epsilon \sum_{k=1}^{N} |n_k|$$

We see that:

$$\frac{1}{2}\gamma \sum_{k=1}^{N} n_k^2 + \frac{\gamma}{2}\sum_{i=1}^{N} 2x_i n_i = \frac{\gamma}{2}\sum_{k=1}^{n}\left(n_k^2 + 2x_k n_k + x_k^2 - x_k^2\right)$$

$$= \frac{\gamma}{2}\sum_{k=1}^{n}\left((n_k + x_k)^2 - x_k^2\right)$$

Since $n_k + x_k = x_{k-1}$, we can easily conclude.

If we assume that the strategy has to be a pure selling program, it means that n_k is necessarily non-negative, thus $|n_k| = n_k$ and $\sum_{k=1}^{N} n_k = X$. Furthermore, if $\tilde{\eta} > 0$, then the expectation of the implementation shortfall is obviously a strictly convex function. The same follows for the function $L(\vec{x}, \lambda)$:

$$L(\vec{x}, \lambda) = \mathbb{E}\left[IS\left(\vec{x}\right)\right] + \lambda\left\{\mathbb{V}\left[IS\left(\vec{x}\right)\right] - v_M\right\}$$

$$= \frac{1}{2}\gamma X^2 + \epsilon X + \frac{\tilde{\eta}}{\tau}\sum_{k=1}^{N}(x_k - x_{k-1})^2 + \lambda\sigma^2 \sum_{k=1}^{N} \tau x_k^2 - \lambda v_M$$

since both λ and $\tilde{\eta}$ are positive.

Since we deal with a problem of convex optimization, we know that there is an equivalence between the existence of the optimum and the Karush-Kuhn-Tucker conditions [4]. It is then possible to find the unique global minimum by setting the partial derivatives of L to zero: for $1 \leq i \leq N-1$

$$\frac{\partial L}{\partial x_i} = 2\tau\left\{\lambda\sigma^2 x_i - \tilde{\eta}\frac{x_{i-1} - 2x_i + x_{i+1}}{\tau^2}\right\} = 0$$

So

$$\frac{1}{\tau^2}\left(x_{i-1} - 2x_i + x_{i+1}\right) = \tilde{\kappa}^2 x_i$$

where

$$\tilde{\kappa}^2 = \frac{\lambda\sigma^2}{\tilde{\eta}}$$

The relationships between x_{i-1}, x_i and x_{i+1} shows that x_i can be written as a linear combination of $e^{\kappa t_i}$ and $e^{-\kappa t_i}$ where $\kappa > 0$ is defined by:

$$\frac{2}{\tau^2}(\cosh(\kappa\tau) - 1) = \tilde{\kappa}^2$$

Indeed, the polynomial related to the above relationship between the terms x_i is:

$$X^2 - 2X + X = (\tilde{\kappa}\tau)^2 X$$

If we consider κ's definition, we can write:

$$e^{\kappa\tau} - 2 + e^{-\kappa\tau} = (\tilde{\kappa}\tau)^2$$

By multiplying either by $e^{\kappa\tau}$ or $e^{-\kappa\tau}$, we can see that those two values are the roots of the polynomial. We can then conclude that x_i can be expressed as a combination of $e^{\kappa t_i}$ and $e^{-\kappa t_i}$.

It is finally possible to find the optimal vector $\vec{x}$ using the initial and final constraints: for $0 \leq i \leq N$

$$x_i = \frac{\sinh(\kappa(T - t_i))}{\sinh(\kappa T)} X$$

Theorem 1 (Optimal Execution Problem)
If we assume that we work with linear market impact functions g and h such that $\eta > \frac{1}{2}\gamma\tau$, and that the strategy is a pure selling program, there exists only one optimal strategy for a given value of $\lambda > 0$, which is given by

$$x_i = \frac{\sinh(\kappa(T - t_i))}{\sinh(\kappa T)} X$$

for $0 \leq i \leq N$, where κ is the unique positive solution of the equation

$$\frac{2}{\tau^2}(\cosh(\kappa\tau) - 1) = \frac{\lambda\sigma^2}{\eta - \frac{1}{2}\gamma\tau}$$

4 Implementation Of The Optimal Strategy

We have implemented such a strategy in a simple case for various values of λ. For the parameters, we used the following numbers:

- $S_0 = 50$,
- $X = 1000000$,
- $T = 50$,
- $\tau = 1$,
- $\sigma = 1$,
- $\gamma = 2.5 \times 10^{-7}$,
- $\epsilon = 0.0625$,
- $\eta = 2.5 \times 10^{-6}$.

Figure 2 displays the trajectory for several values of λ. We can see that, for the highest value of λ, meaning that the trader is risk averse, the strategy has a very short existence: the trader prefers selling all the shares rapidly than holding them for more than a while.

When λ decreases to 0, the trader is less risk averse, and even risk neutral when $\lambda = 0$. The risk neutral strategy is worth noticing: the trader only sells at a constant pace the shares in his or her portfolio between 0 and T.

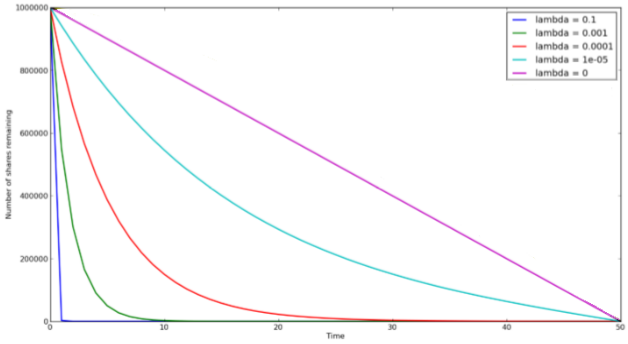

Figure 2: Investment strategy for various values of λ

The implementation illustrates a significant point of our model using linear market impact functions: for a given value of λ, the shares will be sold in exactly the same fashion, not matter what the initial amount X is. For instance, if $\lambda = 0$, the strategy will always be linear. This seems contrary to the intuition: a large amount of shares is objectively less liquid, and then it should be liquidated less rapidly, than a small amount of shares.

This is due to the linearity of the market impact functions g and h. Therefore for large amount of shares, it would be better to use non-linear functions: the costs spawned by a selling decision should increase superlinearly.

Nonetheless if we decide to apply our framework with nonlinear market impact functions, it is no longer possible to obtain explicit solutions as it is in the linear case.

Another solution based on the explicit solutions in the linear case consists in tweaking the parameter η when X increases: the more important X, the higher η.

Indeed, if X is more important, we can choose a higher value for η. So $\tilde{\eta}$ is also higher, $\tilde{\kappa}$ is lower. We can see from the above equations that the effects of a higher value of η are equivalent to those of a lower value for λ. Figure 2 shows that, when λ decreases, the selling strategy is executed less rapidly: this fits with the intuition that a large amount of shares, being less liquid, should be liquidated less rapidly than a small amount of shares.

Conclusion

The difficulty raised by execution cannot be omitted when it comes to implementing live strategies. In this paper we have proposed a first approach of execution questions through market impact considerations. Our model aims at modeling a price which is affected by the actions of a single trader. The trader then needs to find the optimal strategy for a selling program, while taking into account the market impact generated by his or her decisions.

It is possible to work out this so-called optimal execution problem in a simple case, when market impact functions are of a simple, linear form.

Such a model and the optimal execution problem based on it provide a first insight into the issues raised by execution in finance. However many improvements would be necessary in order to make the model more realistic for a real life use. For instance the assumption of linear functions for both permanent and temporary market impact is fairly questionable. Nonetheless the model based on linear functions should not be disregarded too quickly: this model can be reconcile with reality if we keep in mind that the calibration of its parameters is dependent on some of the problem's initial parameters. In this respect the amount of shares that has to be sold plays a pivotal role.

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A propos d’Awalee

Cabinet de conseil indépendant spécialiste du secteur de la Finance.

Nous sommes nés en 2009 en pleine crise financière. Cette période complexe nous a conduits à une conclusion simple : face aux exigences accrues et à la nécessité de faire preuve de souplesse, nous nous devions d’aider nos clients à se concentrer sur l’essentiel, à savoir leur performance.

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